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# Maximum genus and girth of graphs

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**Abstract**

In this paper, a lower bound on the maximum genus of a graph in terms of its girth is established as follows: let $G$ be a simple graph with minimum degree at least three, and let $g$ be the girth of $G$. Then

$$\gamma_M(G) \geqslant \frac{g-2}{2(g-1)}\beta(G) + \frac{1}{g-1} \text{ except for } G = K_4,$$

where $\beta(G)$ denotes the cycle rank of $G$ and $K_4$ is the complete graph with four vertices.


## 1. Introduction

Graphs considered here are all undirected, finite and connected. A graph is called *simple* if it has no multiple edges and loops. For graphical notation and terminology without explanation, we refer to [1].

Let $G = (V(G), E(G))$ be a graph. A set $J \subseteq V(G)$ is called a *nonseparating independent set* (nsis) of $G$, if $J$ is an independent set of $G$ and $G - J$ is connected. The cardinality of a maximum nsis of $G$ is denoted by $z(G)$, the nsis number of $G$. Also, a set $F \subseteq V(G)$ is called a *feedback set* (fvs) of $G$ if the graph $G - F$ is a forest. The cardinality of a minimum fvs of $G$ is denoted by $f(G)$, the fvs number of $G$. The *girth* denoted by $g(G)$, if no confusion, only by $g$, is the length of a shortest cycle in $G$. Let $T$ be a spanning tree of a graph $G$, and let $\xi(G,T)$ be the number of components of $G - E(T)$ with odd number of edges. We define $\xi(G) = \min_T \xi(G,T)$ to be the *Betti deficiency* of $G$, where the minimum is taken over all spanning trees $T$ of $G$.

Recall that the *maximum genus*, denoted by $\gamma_M(G)$, of a graph $G$ is the maximum integer $k$ such that there exists a cellular embedding of $G$ in the orientable surface of


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genus $k$. From the Euler polyhedral equation, we see that the maximum genus of $G$ has the obvious upper bound

$$\gamma_M(G) \leqslant \left\lfloor \frac{\beta(G)}{2} \right\rfloor,$$

where $\beta(G) = |E(G)| - |V(G)| + 1$ is known as the *cycle rank* (or *Betti number*) of the graph $G$. A graph $G$ is called *upper embeddable* if $\gamma_M(G) = \lfloor \beta(G)/2 \rfloor$.

The following basic result, due to Xoung [3], relates the maximum genus to the Betti deficiency of a graph.

**Theorem A.** *Let $G$ be a graph. Then $\gamma_M(G) = (\beta(G) - \xi(G))/2$.*

Maximum genus of graphs has been an interesting topic in the topological graph theory since the introductory investigation by Nordhaus et al. [2]. The relationship between maximum genus and other graph invariants has drawn many a researcher's attention. In particular, the research on the relationship between maximum genus and connectivity of a graph has been active in the recent years, and many papers have appeared to derive a lower bound on the maximum genus of a graph in terms of its connectivity. A previously known result, due to Xoung [3], states that every 4-edge-connected graph $G$ is upper embeddable; that is, its maximum genus arrives at the best upper bound $\lfloor \beta(G)/2 \rfloor$. For a graph with its vertex-(or edge-) connectivity $k < 4$, there exist many such graphs that are not upper embeddable (see [4]), and consequently the papers [5–7] give some tight lower bounds on the maximum genus for the cases $k = 1, 2, 3$, respectively. Detailed results on these can be found in two tables in the paper [7].

In this paper, we study the relationship between the maximum genus and the girth of a graph and present a lower bound on the maximum genus in terms of the girth, yet not the connectivity. The result is described in the abstract of this paper.

The proof of our result is heavily dependent on an equivalence in [11] that for a cubic graph $G$, the maximum genus $\gamma_M(G)$ is equal to $z(G)$, the nsis number of $G$. With the aid of the known bound on $z(G)$ for a cubic simple graph $G$, we first obtain the lower bound on the maximum genus $\gamma_M(G)$ in terms of the girth $g(G)$, then extend this to general graphs so as to arrive at our aim.

## 2. Lemmas and the main results

Before proceeding to the main result, we need some lemmas. The following one is proven in [11].

**Lemma 1.** *Let $G$ be a cubic simple graph. Then $\gamma_M(G) = z(G)$.*

The following result gives the relationship between $f(G)$ and $z(G)$ for a cubic simple graph $G$.

**Lemma 2** (Specknmeyer [9]). *Let $G$ be a cubic simple graph with $n$ vertices. Then* $f(G) = n/2 - z(G) + 1$.

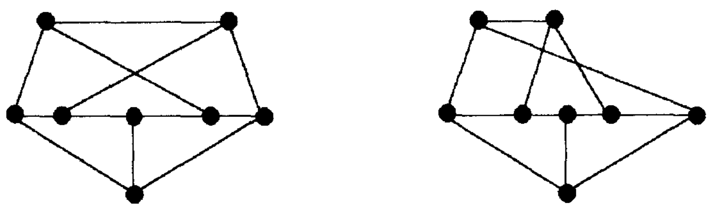
A cubic tree is a tree with each degree either three or one. Denote by $\Psi$ a class of graphs which are derived from a cubic tree by blowing up each degree three vertex to a triangle and attaching the complete graph $K_4$ with one subdivided edge at each degree one vertex. Clearly, it has that $g(G) = 3$ for each $G \in \Psi$.

The following result gives the upper bound on $f(G)$ for a cubic simple graph $G$ in terms of its girth.

**Lemma 3** (Lin and Zhao [10]). *Let $G$ be a cubic simple graph with $n$ vertices and girth $g$. Then*

$$f(G) \leqslant \frac{g}{4(g-1)}n + \frac{g-3}{2g-2}, \text{ except for } G \notin \{K_4, G_1, G_2\} \cup \Psi;$$

*if $G \in \Psi$, then $f(G) = \frac{3}{8}n + \frac{1}{4}$ where $G_1$ and $G_2$ are two special graphs in the following figure*:



$G_1$ $G_2$

Combining Lemmas 1–3, we first obtain the following result for a cubic simple graph.

**Lemma 4.** *Let $G$ be a cubic simple graph with girth $g$. Then*

(1) *if $g \geqslant 4$,*

$$\gamma_M(G) \geqslant \frac{g-2}{2(g-1)}\beta(G) + \frac{3}{2g-2} \text{ except for } G \in \{G_1, G_2\},$$

*where $G_1$ and $G_2$ are the two graphs given in Lemma 3;*

(2) *if $g = 3$, then $\gamma_M(G) \geqslant \frac{1}{4}\beta(G) + \frac{1}{2}$ except for $G = K_4$.*

**Proof.** Let $n$ be the number of the vertices of $G$. By Lemmas 1–3, we get

$$\gamma_M(G) = z(G) = \frac{n}{2} - f(G) + 1.$$

We now prove the conclusions (1) and (2), respectively.

(1) $g \geqslant 4$. Obviously $G \notin \{K_4\} \cup \Psi$ since we see that each graph in $\{K_4\} \cup \Psi$ has girth 3. Thus $G \notin \{K_4, G_1, G_2\} \cup \Psi$ except for $G \in \{G_1, G_2\}$. Therefore, we have

$$\begin{aligned}
\gamma_M(G) &= \frac{n}{2} - f(G) + 1 \\
&\geqslant \frac{n}{2} - \left(\frac{g}{4(g-1)}n + \frac{g-3}{2g-2}\right) + 1 \quad \text{(by the former part of Lemma 3)} \\
&= \frac{g-2}{2(g-1)}\beta(G) + \frac{3}{2g-2} \quad \left(\text{because } G \text{is cubic so} \beta(G) = \frac{n}{2} + 1\right).
\end{aligned}$$

implying conclusion (1) holds.

(2) $g = 3$. We consider this case according as $G \in \Psi$ or $G \notin \Psi$. If $G \in \Psi$, similarly, we have

$$\begin{aligned}
\gamma_M(G) &= \frac{n}{2} - f(G) + 1 \\
&= \frac{n}{2} - \left(\frac{3}{8}n + \frac{1}{4}\right) + 1 \quad \text{(by the latter part of Lemma 3)} \\
&= \frac{1}{4}\beta(G) + \frac{1}{2} \quad \left(\text{because } G \text{is cubic so} \beta(G) = \frac{n}{2} + 1\right).
\end{aligned}$$

If $G \notin \Psi$, since $g(G_1) = g(G_2) = 4$ by observation, then $G \notin \{K_4, G_1, G_2\} \bigcup \Psi$ except for $G = K_4$. By the proof of conclusion (1) and noticing $g = 3$, we immediately get

$$\gamma_M(G) \geqslant \tfrac{1}{4}\beta(G) + \tfrac{3}{4}.$$

In either case, it then follows that

$$\gamma_M(G) \geqslant \min\{\tfrac{1}{4}\beta(G) + \tfrac{1}{2}, \tfrac{1}{4}\beta(G) + \tfrac{3}{4}\} = \tfrac{1}{4}\beta(G) + \tfrac{1}{2},$$

except for $G = K_4$, implying conclusion (2) holds. □

In the following, we shall extend Lemma 4 to general graphs so as to obtain our main result. For this, we introduce the degree lowering technique, which has been successfully used in studying the maximum genus of graphs (for example, see [5,8]).

Let $v$ be a vertex of a simple graph $G$ with the degree $d_G(v) = k \geqslant 4$, and let $u_1, u_2, \ldots, u_k$ be $k$ distinct neighboral vertices of $v$. *Splitting* $v$ *into a path* $P_v$ means as follows: first remove $v$ (also all edges incident with $v$) from $G$; then introduce a $k-2$ length path, denoted by $P_v = x_2x_3 \cdots x_{k-1}$; at last join $x_2$ to both $u_1$ and $u_2$, join $x_{k-1}$ to both $u_{k-1}$ and $u_k$, and join $x_i$ to $u_i$ for $3 \leqslant i \leqslant k-2$. Suppose $G_v$ is the resulting graph.

Observe that $G_v$ is connected and simple if $G$ is, and that $G_v$ has less exactly one vertex of degree $\geqslant 4$ than $G$. On the other hand, we also see that splitting a vertex into a path is converse to contracting edges, and thus contracting all edges of the path $P_v$ in $G_v$ produces the original graph $G$. Besides, there are the following simple but useful relationships between $G$ and $G_v$.

**Lemma 5.** (1) $\beta(G)=\beta(G_v)$ *and* $g(G_v)\geqslant g(G)$; (2) $\gamma_M(G)\geqslant\gamma_M(G_v)$.

**Proof.** By the obtainment of $G_v$, (1) is obvious. Now prove (2). We start with a maximum genus embedding $\rho_M(G_v)$ of the graph $G_v$. Contracting the edges on the path $P_v$ on the embedding $\rho_M(G_v)$ does not decrease the embedding genus and gives an embedding for the graph $G$. That is, the graph $G$ has an embedding of genus $\gamma_M(G_v)$, which implies $\gamma_M(G)\geqslant\gamma_M(G_v)$. □

We are now ready for our main result as follows.

**Theorem** (The main result). *Let G be a simple graph with minimum degree at least three, and let g be the girth of G. Then*

$$\gamma_M(G)\geqslant\frac{g-2}{2(g-1)}\beta(G)+\frac{1}{g-1}\quad \textit{except for } G=K_4.$$

**Proof.** Clearly $g\geqslant 3$ because $G$ is simple. First, consider that $G$ is the cubic graph $G_1$ or $G_2$, where $G_1$ and $G_2$ are the two graphs given in Lemma 3. From observation, $\beta(G_1)=\beta(G_2)=5$, and $g(G_1)=g(G_2)=4$. With the aid of Theorem A, a simple count follows that $\gamma_M(G_1)=\gamma_M(G_2)=2$ because we have $\xi(G_1)=\xi(G_2)=1$ by the definition. This shows that the conclusion is true for $G\in\{G_1,G_2\}$. If $G$ is a cubic graph and $G\notin\{K_4,G_1,G_2\}$, then it easily follows from Lemma 4 that the conclusion is also true. Therefore, the conclusion holds when $G$ itself is a cubic simple graph except for $G=K_4$ (Note however that the theorem here gives a slightly weaker version of Lemma 4 for a cubic simple graph $G$ with the girth $g\geqslant 4$ and $G\notin\{G_1,G_2\}$).

Assume now that $G$ is not a cubic graph. Let $A$ be all vertices of $G$ whose degree is more than three. Then $A\neq\emptyset$. At this time we simply reiterate the degree lowering technique by splitting every vertex in $A$ into a path, eventually produce a connected, simple graph that is cubic. Denote the resulting graph by $G_A$. It is known that $G_A\neq K_4$ because contracting any edge in $K_4$ gives rise to a non-simple graph. Meantime, by continuously applying Lemma 5 as we have done each time, we have

$$\beta(G)=\beta(G_A),\quad g(G)\leqslant g(G_A)\quad\text{and}\quad \gamma_M(G)\geqslant\gamma_M(G_A).$$

Since $G_A$ is a cubic simple graph and $G_A\neq K_4$, by our assertion just before we have

$$\gamma_M(G_A)\geqslant\frac{g(G_A)-2}{2(g(G_A)-1)}\beta(G_A)+\frac{1}{g(G_A)-1}.$$

Now, define a function

$$H(x)=\frac{x-2}{2(x-1)}\beta(G_A)+\frac{1}{x-1}\quad\text{for } x\geqslant 3.$$

It is easy to check that $H(x)$ is an increasing function for $\beta(G_A)\geqslant 2$. Note that

$$\frac{x-2}{2(x-1)}\beta(G_A)+\frac{3}{2x-2}$$

is not an increasing function for $\beta(G_A)\geqslant 2$. This is the reason why we need to use a weak lower bound

$$\gamma_M(G_A)\geqslant \frac{g(G_A)-2}{2(g(G_A)-1)}\beta(G_A)+\frac{1}{g(G_A)-1}$$

instead of the stronger lower bound

$$\gamma_M(G_A)\geqslant \frac{g(G_A)-2}{2(g(G_A)-1)}\beta(G_A)+\frac{3}{2g(G_A)-2}$$

derived in Lemma 4. We see that if any simple graph $G$ has minimum degree at least 3, then $\beta(G)\geqslant 2$. Therefore, we have the following:

$$\begin{aligned}
\gamma_M(G) &\geqslant \gamma_M(G_A)\\
&= H(g(G_A))\\
&\geqslant H(g) \quad \text{(because } H(x) \text{ is an increasing function for } \beta(G_A)\geqslant 2,\\
&\qquad\qquad g(G_A)\geqslant g, \text{ and } \beta(G_A)\geqslant 2)\\
&= \frac{g-2}{2(g-1)}\beta(G_A)+\frac{1}{g-1}\\
&= \frac{g-2}{2(g-1)}\beta(G)+\frac{1}{g-1} \quad \text{(because } \beta(G_A)=\beta(G)),
\end{aligned}$$

which completes the proof. □

In [5], it is proven that $\frac{1}{4}\beta(G)$ is a tight lower bound on the maximum genus for any simple graph $G$ with minimum degree at least 3. In [6,7], it is also proven that $\frac{1}{3}\beta(G)$ is a tight lower bound on the maximum genus for a 2-vertex-connected (also 2-edge-connected) simple graph with minimum degree at least 3, even for a 3-vertex-connected (also 3-edge-connected) simple graph. The following corollary gives a slight improvement on the main result in [5], and also indicates that $\frac{1}{3}\beta(G)+\frac{1}{2}$ is a lower bound on the maximum genus under the condition that $G$ has no triangles, i.e., $g(G)\geqslant 4$, or $G$ is bipartite, yet not the connectivity.

**Corollary 1.** *Let $G$ be a simple graph with minimum degree at least three. Then*
(1) $\gamma_M(G)\geqslant\frac{1}{4}\beta(G)+\frac{1}{2}$ *except for* $G=K_4$;
(2) *if $G$ has no triangles,* $\gamma_M(G)\geqslant\frac{1}{3}\beta(G)+\frac{1}{3}$.

**Proof.** Note that the function

$$H(x)=\frac{x-2}{2(x-1)}\beta(G)+\frac{1}{x-1}, \quad x\geqslant 3$$

is an increasing function for $\beta(G)\geqslant 2$. Note also that $\beta(G)\geqslant 2$ and $g(G)\geqslant 3$ for any simple graph $G$ with minimum degree at least 3. Hence, by our theorem

$$\gamma_{\mathrm{M}}(G)\geqslant H(g)\geqslant H(3)=\tfrac{1}{4}\beta(G)+\tfrac{1}{2},$$

implying (1) holds.

If $G$ has no triangles, obviously $g\geqslant 4$. Similarly, we have

$$\gamma_{\mathrm{M}}(G)\geqslant H(g)\geqslant H(4)=\tfrac{1}{3}\beta(G)+\tfrac{1}{3},$$

implying (2) holds. □

We end this section with a result on the upper embeddability of a graph in connection with its girth, which may be viewed as a generalization in [11].

**Corollary 2.** *Let $G$ be a graph with minimum degree at least three, and let $g$ be its girth. Then* $\lim_{g\to\infty}\gamma_{\mathrm{M}}(G)/\lfloor(\beta(G))/2\rfloor=1$.

**Proof.** We may assume that $g>3$ and thus that $G$ is simple. By our theorem above and noticing $\gamma_{\mathrm{M}}(G)\leqslant\lfloor\beta(G)/2\rfloor$, it immediately follows that

$$1\geqslant\gamma_{\mathrm{M}}(G)\Big/\left\lfloor\frac{\beta(G)}{2}\right\rfloor\geqslant\frac{g-2}{g-1}+1\Big/\left((g-1)\left\lfloor\frac{\beta(G)}{2}\right\rfloor\right).$$

Thus, $\lim_{g\to\infty}\gamma_{\mathrm{M}}(G)/\lfloor(\beta(G))/2\rfloor=1$, completing the proof. □

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